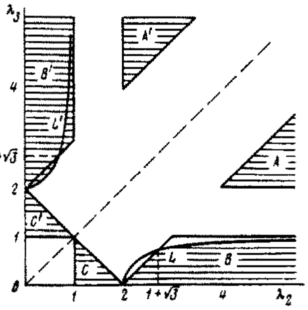

In the solutions corresponding to cases a), and c), the pressure falls with distance from the centre of the gas ellipsoid. These solutions do not have any singularities, since $d^3d^{\cdot\cdot} = C_3 > 0$. In case b) the cloud moves due to the action of the external pressure, which varies with time as given by (1.3).

As can be seen from (4.4), C_3 vanishes along the line L, specified by the equation

$$\lambda_2^2 + \lambda_3^2 - \lambda_2^2\lambda_3^2 - 4 = 0$$

and splits the region B into two subregions (see the figure). For a point lying in region B above the line L, $C_3 < 0$, i.e. a singularity $(d = 0)$ must necessarily occur in the solutions corresponding to it, namely, a state in which the volume of the cloud vanishes, while the density and pressure become infinite. For points lying below L, $C_3 > 0$, i.e. d does not vanish; the rotation and internal vorticity of the gas cloud prevents its collapse.

A complete picture in the (λ_2, λ_3) plane is obtained after symmetric reflection of regions A, B, and C in the straight line $\lambda_2 = \lambda_3$, and of the line L into regions A', B', and C' and the line L' respectively.

REFERENCES

1. RIEMANN B.O., The motion of a uniform liquid ellipsoid, in: Complete Works, Gostekhizdat, Moscow-Leningrad, 1948.
2. CHANDRASEKAR S., Ellipsoidal figures of equilibrium, Mir, Moscow, 1973.
3. SEDOV L.I., Integration of the one-dimensional motion of a gas, Dokl. Akad. Nauk SSSR, 90, 1953.
4. SEDOV L.I., The methods of similitude and dimensions in mechanics (Metody podobiya i razmernosti v mekhanike), NAUKA, Moscow, 1981.
5. OVSYANIKOV L.V., A new solution of the equations of hydrodynamics, Dokl. Akad. Nauk SSSR, 111, 1956.
6. DYSON J.F., Dynamics of a spinning gas cloud. J. Math. and Mech., Vol.18, No.1, 1968.
7. ANISIMOV S.I. and LYSIKOV YU.I., The expansion of a gas cloud in a vacuum, Prikl. Matem. Mekhan. 34, 1970.
8. OSIN A.I. and POSLAVSKII S.A., Motion with uniform deformation in gas dynamics, Vestnik MGU, Matem. Mekhan. No.6, 1981.
9. BOGOYAVLENSKII O.I., Methods of the qualitative theory of dynamic systems in astrophysics and gas dynamics (Metody kachestvennoi teorii dinamicheskikh sistem v astrofiziki i gazovi dinamike), NAUKA, Moscow, 1980.

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RELATIVISTIC PRANDTL- MEYER FLOW*

N.I. KOLOSNITSYN and K.P. STANYUKOVICH

The exact solution of the equations of relativistic gas dynamics describing plane steady-state flow, depending only on the angular variable, is investigated. The well-known Prandtl-Meyer solution is obtained in the non-relativisitic limit.

The problem of constructing relativistic Prandtl-Meyer flow has been considered in /1—3/. A solution was obtained in /1/, by direct integration of the equations, describing the limiting case of ultrarelativistic flow. In /2, 3/, to obtain relativistic Prandtl-Meyer flow, the method of replacement of variables proposed in /4/ was used, by means of which the equations of relativistic hydrodynamic were reduced to Newtonian form for a certain auxiliary gas with a variable isentropy index. Using this

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method ultrarelativistic flow was considered in /2/, and a solution was obtained in /3/ for a one-component ideal gas with the equation of state /5/. The use of the variable proposed in /4/ does not remove the problem of integration, so that in a number of cases it may be preferable to integrate the initial equations of relativistic gas dynamics directly. We used this method here, i.e., by direct integration of the equations of relativistic gas dynamics we obtain an accurate solution describing Prandtl-Meyer flow of an ideal gas with an arbitrary isentropy index.

The equations of relativistic gas dynamics for isoentropic flows can be written in the generally accepted notation as follows /6/:

$$\frac{\partial}{\partial x^k}\left(\sqrt{-g}\,\frac{u^k}{V}\right)=0,\quad u^k\left(\frac{\partial W u_i}{\partial x^k}-\frac{\partial W u_k}{\partial x^i}\right)=0 \tag{1}$$

Consider plane steady-state flow, which depends solely on the angular variable θ. Changing in (1) to a cylindrical system of coordinates and retaining the dependence solely on θ, we arrive at the following set of equations:

$$w=\frac{du}{d\theta},\quad \frac{u}{V\beta}+\frac{d}{d\theta}\left(\frac{w}{V\beta}\right)=0,\quad \frac{W}{\beta}=\mathrm{const} \tag{2}$$

to which the following relation is connected:

$$\beta^2=1-(u^2+w^2)/c^2 \tag{3}$$

Here u and w are the radial and tangential components of the velocity. The set of equations (2)—(3) is closed by the isentropy equation $pV^k=\mathrm{const}$.

The second equation (2) is equivalent to the equation

$$u+\frac{dw}{d\theta}-w\left(\frac{d\ln V}{d\theta}+\frac{d\ln\beta}{d\theta}\right)=0 \tag{4}$$

Introducing the velocity of sound a and taking the third equation of (2) into account, we can convert (4) to the form

$$u+\frac{dw}{d\theta}+w\left(1-\frac{a^2}{c^2}\right)\frac{c^2}{a^2}\frac{d\ln W}{d\theta}=0,\quad \frac{a^2}{c^2}=-\left.\frac{\partial\ln W}{\partial\ln V}\right|_\sigma \tag{5}$$

On the other hand, by differentiating (3) and taking the third equation of (2) into account, we obtain

$$w\left(u+\frac{dw}{d\theta}\right)+c^2\left(1-\frac{u^2+w^2}{c^2}\right)\frac{d\ln W}{d\theta}=0 \tag{6}$$

Multiplying (5) by w, subtracting (6) from the result, and assuming that $d\ln W/d\theta\neq 0$, we obtain

$$w^2=a^2\left(1-\frac{u^2}{c^2}\right) \tag{7}$$

The Bernoulli equation (the third equation of (2)), taking into account the relationship between the enthalpy and the velocity of sound

$$\frac{c^2}{W}=1-\frac{1}{k-1}\frac{a^2}{c^2} \tag{8}$$

can be written in the form

$$\left(1-\frac{u^2+w^2}{c^2}\right)\left(1-\frac{1}{k-1}\frac{a^2}{c^2}\right)=1-\frac{q^2}{c^2} \tag{9}$$

(q is the maximum flow velocity, which is reached at the liquid-outflow boundary when $a=0$).

Equations (8) and (9) in terms of u^2 and w^2 form a set of linear algebraic equations. Solving them we obtain

$$\frac{u^2}{c^2}=1-F^2(a),\quad w^2=a^2F^2(a) \tag{10}$$
$$F^2(a)=\left(1-\frac{q^2}{c^2}\right)\left(1-\frac{a^2}{c^2}\right)^{-1}\left(1-\frac{1}{k-1}\frac{a^2}{c^2}\right)^{-2}$$

Taking these relations into account, we obtain from the first equation of (2) the quadrature

$$\theta+\theta_0=\pm c\int\frac{F'(a)\,da}{a\sqrt{1-F^2(a)}} \tag{11}$$

which defines the relationship between a and θ. Further, using (10) we obtain $u(\theta)$ and $w(\theta)$, and then all the remaining characteristics of the relativistic Prandtl-Meyer flow.

Consider some limiting cases.

For non-relativisitic flow $(q/c \ll 1,\ a/c \ll 1)$, it follows from (11) that

$$\theta+\theta_0=\pm\sqrt{\frac{k+1}{k-1}}\arcsin\left(\sqrt{\frac{k+1}{k-1}}\frac{a}{q}\right)$$

In non-relativistic flow $a^2=w^2$, and hence, changing from the inverse to the direct functions, we obtain

$$w=\pm\sqrt{\frac{k-1}{k+1}}q\sin\left[\sqrt{\frac{k-1}{k+1}}(\theta+\theta_0)\right],\qquad u=\mp q\cos\left[\sqrt{\frac{k-1}{k+1}}(\theta+\theta_0)\right] \tag{12}$$

i.e., we obtain the well-known Prandtl-Meyer solution /7/.

To change to the ultrarelativistic limit, we must integrate by parts in (11). We obtain

$$\theta+\theta_0=\pm\frac{c}{a}\arcsin F(a)\pm c\int\arcsin F(a)\frac{da}{a} \tag{13}$$

In an ultrarelativistic gas, the velocity of sound is constant, so that the integral in (13) must be understood in the Lebesgue sense. This integral is taken over a bounded and measurable function in a set of measure zero and is equal to zero. Finally, using (10) and the well-known value of the velocity of sound $a=\sqrt{k-1}c$ we obtain

$$w=\pm\sqrt{k-1}c\sin[\sqrt{k-1}\ (\theta+\theta_0)],\quad u=\mp c\cos[\sqrt{k-1}\ (\theta+\theta_0)] \tag{14}$$

which agrees with the corresponding solutions obtained for ultrarelativistic flow in /1, 2/.

In conclusion, we will obtain a solution for ultrarelativistic flow by the method used in /4/. In the variables employed by Shikin /4/, the relativistic equations for an ideal gas with isentropy index k can be reduced to the equations of Newtonian gas dynamics for a gas with a variable isentropy index k_*, which, as can be shown, is equal to

$$k_*=k\frac{W}{c^2}\left[(2-k)\frac{W}{c^2}+(k-1)\right]^{-1} \tag{15}$$

In the ultrarelativistic case $W/c^2\gg 1$, and from (15) we have

$$k_*=k/(2-k)=\text{const}$$

The effective isentropy index is constant, and so we can use the formulas of ordinary relativistic Prandtl-Meyer flow (12), completing them by making the replacement of variables

$$k\to k_*=k/(2-k),\ q\to c$$

Equations (12) then reduce to (14) for ultrarelativistic flow.

REFERENCES

1. ARYNOV A. and SHABALIN V.D., Relativistic Prandtl-Meyer flow, in: Plane-parallel and axisymmetric flow of gases and liquids (Ploskoparallel'noe i osesimmetrichnoe techenie gazov i zhidkostei), ILIM, Frunze, 1966.
2. KÖNIGL A., Relativistic gas dynamics in two dimensions. Phys. Fluids, Vol.23, No.6, 1980.
3. CHIU H.H., Relativistic gas dynamics. Phys. Fluids, Vol.16, No.6,1973.
4. SHIKIN I.S., The general theory of stationary motion in relativistic hydrodynamics, Dokl. Akad. Nauk SSSR, 142, 1962.
5. SYNGE D.L., Relativistic gas, Atomizdat, Moscow, 1960.
6. FRANKL F.I., Collected papers on gas dynamics (Izbrannye trudy po gazovoi dinamike), NAUKA, Moscow, 1978.
7. STANYUKOVICH K.P., Non-steady-state flow of a continuous medium (Neustranovivshiesya techeniya sploshnoi sredy), Gostekhizdat, Moscow, 1955.

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